International Journal of Advanced Research in Engineering and Technology (IJARET)
Volume 6, Issue 3, March, 2015, pp. 62-69, Article ID: IJARET_06_03_007
Available online at
http://www.iaeme.com/IJARET/issues.asp?JTypeIJARET&VType=6&IType=3
ISSN Print: 0976-6480 and ISSN Online: 0976-6499

ITERATIVE METHODS FOR THE SOLUTION OF SADDLE POINT PROBLEM

Ndzana Benoît
Senior Lecturer, National Advanced School of Engineering,
University of Yaounde I, Cameroon

Biya Motto
Frederic, Senior Lecturer, Faculty of Sciences,
University of Yaounde I, Cameroon

Lekini Nkodo Claude Bernard
P.H.D. Student; National Advanced School of Engineering,
University of Yaounde I, Cameroon

ABSTRACT

Some new iterative methods for numerical solution of mixed finite element approximation of Stokes problem are presented. The idea is the use of proper preconditioning for the conjugate gradient algorithm. A particular case gives a variant of the Arrow-Hurwicz method.

Keywords: Iterative method, Stokes problem, Arrow-Hurwicz method

Cite this Article: Benoît, N., Motto, B. and Bernard, L. N. C. Iterative Methods for the Solution of Saddle Point Problem *International Journal of Advanced Research in Engineering and Technology*, **6**(3), 2015, pp. 62-69.
http://www.iaeme.com/IJARET/issues.asp?JType=IJARET&VType=6&IType=3

1. STATEMENT OF THE PROBLEM

Let us consider a polygonal domain $\Omega \subset R^n$ (n=2 or 3) of regular boundary $\partial\Omega = \Gamma$.

Let us denote $V = \{v \in (H1(\Omega))^n\}, Q = L^2(\Omega), \epsilon(v) = \left(\epsilon_{ij}(v)\right)_{1\leq i,j\leq n}$ and

$$\epsilon_{ij}(v) = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] \tag{1.1}$$

The Stokes problem for fluid flow is

$$\begin{cases} -v\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\epsilon_{ij}(u)+(\nabla p)_i=f_i, 1\le i\le n \; in \; \Omega, \\ \nabla.u=0 \; in \; \Omega \\ u\in V, p\in Q, \end{cases} \tag{1.2}$$

Where $f=(f_1,\dots,f_n)$ is a given function inQ^n and v the viscosity of the fluid. We look for a velocity vector u and a static pressure p. The problem (1.1) – (1.2) can be formulated as the saddle point problem.

$$Inf_{v\in V}\, Sup_{q\in Q}\, L(v,q)=Inf_{v\in V}\, Sup_{q\in Q}\,(J(v)-(q,\nabla.v)) \tag{1.3}$$

Where $J(v)=\frac{1}{2}v\int_{\Omega}\epsilon(v):\epsilon(v)dx-\int_{\Omega}f.v\,dx=\frac{1}{2}a(v,v)-(f,v)$

With the condition $\nabla.v=0$ we obtain

$$a(v,v)=v\int_{\Omega}\epsilon(v):\epsilon(v)dx=v\int_{\Omega}\nabla v:\nabla v\,dx \tag{1.4}$$

Taking the equilibrium condition for (1.3) gives:

$$\begin{cases} v\int_{\Omega}\nabla u:\nabla v\,dx-\int_{\Omega}f.v\,dx-\int_{\Omega}p\nabla.v\,dx=0 \;\; \forall\, v\in V, \\ \int_{\Omega}\nabla.u\,q\,dx=0\;\forall\, q\in Q. \end{cases} \tag{1.5}$$

This is clearly a variational formulation of (1.1) – (1.2). Let $V_h\subset V$ and $Q_h\subset Q$ be two families of finite element spaces approximations V and Q respectively, A and B the discrete operators approaching $(-v\Delta)$ and ∇ respectively in these approximation. The discretized Stokes problem is to find $(u,p)\in V_h\times Q_h$, solution of the following symmetric and indefinite system

$$\begin{cases} Au+B^t p=f \; in \; \Omega \\ Bu=0 \; in \; \Omega \end{cases} \tag{1.6}$$

Denoting: A=$\begin{bmatrix} A & B^t \\ B & 0 \end{bmatrix}$, $X=\begin{bmatrix} u \\ p \end{bmatrix}$ and $F=\begin{bmatrix} f \\ 0 \end{bmatrix}$.

The problem (1.6) takes the following form

$$\mathrm{A}X=F, \tag{1.7}$$

A is indefinite symmetric matrix with a particular structure. The discrete primal problem obtained by restricting to the divergence-free subspace can be written as

$$Inf_{v\in V_{oh}}\frac{1}{2}(A_0 v,v)-(f_0,v) \tag{1.8}$$

$$V_{oh}=\{v\in V_h | Bv=0\} \tag{1.9}$$

Where A_0 and f_0 are respectively the restriction of A and f to V_{oh}, (1.8) is equivalent to the minimization problem.

The discrete dual problem is

$$Inf_{q \in Q_h} \frac{1}{2}(Dq, q) - (f^*, q) \tag{1.10}$$

Where $D = BA^{-1} B^t$ and $f^* = BA^{-1} f$, this problem is also a standard quadratic problem although $D = BA^{-1}B^t$ cannot be computed explicitly as A^{-1} is a full matrix. We shall develop algorithms taking into account the special structure of A to obtain a general family of iterative methods some of which will be explicited and analysed. Before doing so we recall the application of classical optimization techniques to the primal and dual problems (1.8) and (1.10).

2. CLASSICAL SOLUTION METHODS

2. 1. Primal Problem

The Navier-Stokes and Stokes problem require a preconditioner yielding a divergence-free solution and a divergence-free descent direction at each iteration [2, 7]. An efficient example of this preconditioner can be built throught the block relaxation method, the advantage of which is to solve a discretized problem in a small subregion. We have described in previous paper (cf [1], [4], [5]) such method. Let us denote S this preconditioning operator, the P.C.G. algorithm becomes.

2.1.1. Alg. 2.1:

Step 1: Select an initial divergence-free solution u^0
Step 2: Compute the divergence-free descent direction

$$z^n = S^{-1}g^n \tag{2.2}$$

Where $g^n = Au^n - f$, and compute

$$\Phi^n = z^n + \beta_n \Phi^{n-1} \text{ so that } A\Phi^n \perp \Phi^{n-1} \tag{2.3}$$

Step 3: Compute α_n, u_{n+1} and g_{n+1} by

$$\begin{aligned} u_{n+1} &= u^n - \alpha_n \Phi^n \\ g_{n+1} &= Au_{n+1} - f \end{aligned} \tag{2.4}$$

α_n defined by the condition $g_{n+1} \perp \Phi^n$.

This is in fact the usual P.C.G method on the divergence-free subspace.

2.2. Dual Problem

We remember in the following the C.G Uzawa algorithm for resolution of the dual problem.

2.2.1. Alg. 2.2:

Step 1: Let $p^0 \in Q_h, u^0 = A^{-1}(f - B^t p^0)$, suppose u^n known.
Step 2: Compute the descent direction

$z^n = (z_u^n, z_p^n) = (-A^{-1}B^tBu^n, Bu^n)$ and compute

$$\Phi^n = z^n + \beta_n \Phi^{n-1} = (\phi_u^n, \Phi_p^n)$$

$$\beta_n = \frac{|Bu^n|^2}{|Bu^{n-1}|^2} \tag{2.5}$$

Step 3: Compute α_n, u_{n+1} and p_{n+1} by

$$\alpha_n = -\frac{|Bu^n|^2}{(Bu^n, B\Phi_u^n)} \tag{2.6}$$

$$\begin{cases} u^{n+1} = u^n - \alpha_n \Phi_u^n \\ p^{n+1} = p^n - \alpha_n \Phi_p^n \end{cases} \tag{2.7}$$

This algorithm requires to solve exactly the linear system $Au = f - B^t p$ at each iteration. If we solve approximatively this system, we obtain Arrow-Hurwicz's algorithm to find the saddle point. The steepest descent method is obtained for $\beta_n = 0$; for $\alpha = 0$ constant we obtain the Uzawa algorithm.

3. GENERAL FORMULATION

The principal idea of this method is to combine the two algorithms Alg. 2.1 and alg. 2.2 for solving mutually the primal and the dual problems. The system (1.7) is indefinite, the standard C.G method yields a divergent iterative method. However with a good preconditioner and proper descent direction we can obtain a convergent iterative method, which coincides with a variant of Arrow-Hurwicz algorithm (cf. [3]). The next algorithm is interesting when the projection on a divergence-free subspace is difficult or very expensive with the preconditioner used. Let S be a preconditioning operator of A (cf. [1]), R^n and R^n are the residuals respectively defined by

$$R^n = \begin{pmatrix} r_u^n \\ 0 \end{pmatrix} \quad \text{where } r_u^n = Au^n + B^t p^n - f \tag{3.1}$$

and

$$R^n = \begin{bmatrix} 0 \\ r_p^n \end{bmatrix} \text{where } r_p^n = Bu^n \tag{3.2}$$

The step descent directions are defined as solutions of the following problems

$$Sz^n = R^n \tag{3.3}$$

$$Sz^n = R^n \tag{3.4}$$

Those directions are defined to minimize respectively the resuduals of the primal and the dual problems.

3.1. G. "Primal-Dual" Algorithm

Step 1: Select an initial solution(u^0, p^0).

Step 2: Solve $Sz^n = R^n$

α_n is defined by a condition in order to minimize the residual R^n,

$$r_u^{\frac{n+1}{2}} = u^n - \alpha_n z_u^n \tag{3.5}$$

Compute $u^{\frac{n+1}{2}}$ and $p^{\frac{n+1}{2}}$

$$\begin{cases} u^{\frac{n+1}{2}} = u^n - \alpha_n z_u^n \\ p^{\frac{n+1}{2}} = p^n - \alpha_n z_p^n \end{cases} \tag{3.6}$$

So that

$$\begin{cases} r_u^{\frac{n+1}{2}} = Au^{\frac{n+1}{2}} + B^t p^{\frac{n+1}{2}} - f \\ r_p^{\frac{n+1}{2}} = Bu^{\frac{n+1}{2}} \end{cases} \tag{3.7}$$

Step 3: Solve $Sz^n = R_{\frac{n+1}{2}}$

α_n defined by a condition in order to minimize the residual$R_{\frac{n+1}{2}}$, i.e.

$$r_p^{n+1} \perp z_p^n \tag{3.8}$$

Compute u_{n+1} and p^{n+1}

$$\begin{cases} u^{n+1} = u^{\frac{n+1}{2}} - \alpha_n z_u^n \\ p^{n+1} = p^{\frac{n+1}{2}} - \alpha_n z_p^n \end{cases} \tag{3.9}$$

So that

$$\begin{cases} r_u^{n+1} = Au^{n+1} + B^t p^{n+1} - f \\ r_p^{n+1} = Bu^{n+1} \end{cases} \tag{3.10}$$

3.2. Primal-Dual Algorithm

Since every one of the steps 2 and 3 correspond to the gradient algorithm for minimization of the residual for a quadratic problem, we can introduce the orthogonality relations associated to the primal problem matrix A, and the approximate dual problem matrix $D = BS^{-1}B^t$, we obtain the following algorithm.

3.2.1. Alg. 3.1.

Step 1: Select an initial solution (u^0, p^0)

Step 2: Compute β_n and Φ^n by a condition

$$\begin{cases} r_u^{\frac{n+1}{2}} \perp \Phi_u^{n-1} \\ \Phi^n = z^n + \beta_n \Phi^{n-1} \end{cases} \tag{3.11}$$

Compute α_n, $u_{\frac{n+1}{2}}$ and $p_{\frac{n+1}{2}}$; α_n defined by a condition

$$r_u^{\frac{n+1}{2}} \perp \Phi_u^n \tag{3.12}$$

$$\begin{cases} u^{\frac{n+1}{2}} = u^n - \alpha_n \Phi_u^n \\ p^{\frac{n+1}{2}} = p^n - \alpha_n z_p^n \end{cases} \tag{3.13}$$

So that

$$\begin{cases} r_u^{\frac{n+1}{2}} = Au^{\frac{n+1}{2}} + B^t p^{\frac{n+1}{2}} - f \\ r_p^{\frac{n+1}{2}} = Bu^{\frac{n+1}{2}} \end{cases} \tag{3.14}$$

Step 3: Solve $Sz^n = R_{\frac{n+1}{2}}$

Compute β_n and Φ^n by a condition

$$r_p^{n+1} \perp \Phi_p^{n-1} \tag{3.15}$$

$\Phi^n = z^n + \beta_n \Phi_{n-1}$

Compute α_n, u_{n+1}, p_{n+1}. α_n defined by a condition

$$r_p^{n+1} \perp \Phi_p^n \tag{3.16}$$

$$u^{n+1} = u^{\frac{n+1}{2}} - \alpha_n \Phi_u^n$$

$$p^{n+1} = p^{\frac{n+1}{2}} - \alpha_n \Phi_p^n$$

So that

$$r_u^{n+1} = Au^{n+1} + B^t p^{n+1} - f$$

$$r_p^{n+1} = Bu^{n+1}$$

Doing some iterations of Step 2 before moving to Step 3, we obtain a convergence of the primal variable corresponding to a minimum in v of the lagrangien $L(v, q)$ and we pull back on Uzawa's method (Alg. 2.2).

On the contrary if we obtain convergence of the approximate dual variable and a divergence-free solution after some iteration of Step 3, the whole process can be reduced to the primal algorithm (Alg. 2.1). With a good preconditioner we can obtain easily the divergence-free condition and we find once again the alg. 2.1. If the

preconditioner yields rapidly the convergence of the primal problem we find once again the Uzawa algorithm. The study of the convergence depends of the preconditioner (cf. R. Aboulaich [1]). The convergence is illustrated in Figure 1. In the following we present a particular example of preconditioning operator and we find a variant of Arrow-Hurwicz algorithm.

3.3. A Particular Case of Preconditioning

Let us denote $S = \begin{bmatrix} S & B^t \\ 0 & 1 \end{bmatrix}, R^n = \begin{bmatrix} r_u^n \\ 0 \end{bmatrix}$ and $R^n = \begin{bmatrix} 0 \\ r_p^n \end{bmatrix}$

The G. "P-D" algorithm becomes:

Alg. 3.2

Step 1: The same

Step 2: Solve

$$\begin{cases} Sz_u^n + B^t z_p^n = r_u^n \\ z_p^n = 0 \end{cases} \quad => z_u^n = S^{-1} r_u^n \tag{3.19}$$

And we compute

$$\alpha_n = \frac{(r_u^n, z_u^n)}{(Az_u^n, z_u^n)}, \tag{3.20}$$

$$\begin{cases} u^{\frac{n+1}{2}} = u^n - \alpha_n z_u^n \\ p^{\frac{n+1}{2}} = p^n \end{cases} \tag{3.21}$$

Step 3: Solve

$$\begin{cases} Sz_n^n + B^t z_p^n = 0 \\ z_p^n = r_p^n \end{cases} \tag{3.22}$$

And we compute

$$\alpha_n = \frac{(Bu^n, z_p^n)}{(Bz_u^n, z_p^n)}, \tag{3.23}$$

$$\begin{cases} u^{n+1} = u^{\frac{n+1}{2}} - \alpha_n z_u^n \\ p^{n+1} = p^{\frac{n+1}{2}} - \alpha_n z_p^n \end{cases} \tag{3.24}$$

We obtain

$$u^{\frac{n+1}{2}} = u^n - \alpha_n S^{-1} r_u^n \tag{3.25}$$

$$u^{n+1} = u^{\frac{n+1}{2}} + \alpha_n S^{-1} B^t r_p^n \tag{3.26}$$

$$p^{n+1} = p^n - \alpha_n r_p^n \tag{3.27}$$

We find a variant of Arrow-Hurwicz algorithm. A must practical variante of the previous algorithm is obtained to compute parallel the two descente directions z and z, the algorithm becomes:

$$\begin{cases} u^{n+1} = u^n - \alpha_n S^{-1} r_u^n - \alpha_n S^{-1} B^t B u^n, \\ \qquad p^{n+1} = p^n + \alpha_n B u^n. \end{cases} \tag{3.28}$$

Also we can use the direction $g^n = z^n + \gamma z^n$ where

$z^n = \begin{bmatrix} z_u^n \\ 0 \end{bmatrix}$ and $z^n = \begin{bmatrix} S^{-1} B^t B u^n \\ -B u^n \end{bmatrix}$.

The iteration (4.12) becomes

$$\begin{cases} u^{n+1} = u^n - \alpha_n g_u^n, \\ p^{n+1} = p^n - \alpha_n B u^n, \end{cases} \tag{3.29}$$

α_n defined to minimize the residual $R^n = \begin{bmatrix} r_u^n \\ r_p^n \end{bmatrix}$, so that $R^{n+1} \perp g^n$, and we have the possibility to use the P.C.G algorithm for the saddle point problem, with the recherché direction $\psi^n = g^n + \beta_n \psi^{n-1}$ such that $(A\psi^{n+1}, \psi^n) = 0$. The problem is to define properly the parameter γ, when γ is constant we obtain convergence of the algorithm but is difficult to compute the optimal parameter.

REFERENCES

[1] Aboulaîch, R. Ph. D, Thesis, Département de Mathématiques, statistique et actuariat, Université Laval, Québec. Canada.

[2] Aboulaîch, R., Fortin, M., Robichaud, M. and Tanguy, P. Several Iterative Schemes for the Solution of the Navier-Stokes Equations.

[3] Ekeland, I. and Temam, R. Analyse convexe et problems variationnels.

[4] Fortin, M. An Iterative Method for Finite Element Fluid Flow Simulation. Fourth Int. Symp. On Num. Meth in Eng., Atlanta, March 1986.

[5] Aboulaîch, R. and Fortin, M. Une méthode du G.C.P pour la resolution numérique des équations de Navier Stokes. First International Conference in Africa computer methods and water ressources 1988.

[6] Fortin, M. and Glowinski, R. Méthodes de lagrangien augmenté. Dunod, Paris 1982.

[7] Temam, R. Navier-Stokes Equations. Studies in Math. and its Applications, Lions, J. L., Papanicolaou, G. and Rockafellar, R.T. ed. North Holland, 1977.